BASICS OF SAR POLARIMETRY II

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Fundamentals of SAR Polarimetry

Abstract A comprehensive overview of the basic principles of radar polarimetry is presented. The relevant fundamental field equations are first provided. The importance of the propagation and scattering behavior in various frequency bands, the electrodynamic foundations such as Maxwell's equations, the Helmholtz vector wave equation and especially the fundamental laws of polarization will first be introduced: The fundamental terms which represent the polarization state will be introduced, defined and explained. Main points of view are the polarization Ellipse, the polarization ratio, the Stokes Parameter and the Stokes and Jones vector formalisms as well as its presentation on the Poincaré sphere and on relevant map projections. The Polarization Fork descriptor and the associated van Zyl polarimetric power density and Agrawal polarimetric phase correlation signatures will be introduced also in order to make understandable the polarization state formulations of electromagnetic waves in the frequency domain. The polarization state of electromagnetic waves under scattering conditions i.e. in the radar case will be described by matrix formalisms. Each scatterer is a polarization transformer; under normal conditions the transformation from the transmitted wave vector to the received wave vector is linear and this behavior, principally, will be described by a matrix called scattering matrix. This matrix contains all the information about the scattering process and the scatterer itself. The different relevant matrices, the respective terms like Jones Matrix, S-matrix, Müller M-matrix, Kennaugh K-matrix, etc. and its interconnections will be defined and described together with change of polarization bases transformation operators, where upon the optimal (Characteristic) polarization states are determined for the coherent and partially coherent cases, respectively. The lecture is concluded with a set of simple examples.

4. Polarimetric Radar Optimization for the Coherent Case

The optimization of the scattering matrices, derived for the mono-static case is separated into two distinct classes. The first one, dealing with the optimization of $[S]$, $[G]$, and $[K]$, for the coherent case results in the formulation of '*Kennaugh's target matrix characteristic operator and tensorial polarization fork*' and the associated renamed '*Huynen Polarization Fork*' concept plus the *'co/cross-polarization power density plots'* and the 'co/cross-polarization phase correlation plots', also known as the van Zyl [79, 71] and the Agrawal plots[78, 90], respectively, in the open literature. The second one, presented in Chapter 5, deals with the optimization for the partially polarized case in terms of the '*lexicographic and the Pauli-based covariance matrices,* $[C_L]$ *and* $[C_P]$*, respectively*', as introduced in Sections 3.7 to 3.10, resulting in the '*Cloude target decomposition theorems*' and the Cloude-Pottier [27, 57, 58] supervised and unsupervised '*Polarimetric Entropy* H*, Anisotropy* A*, and* $\overline{\alpha}$*-Angle Descriptors*'. In addition, the '*polarimetric contrast optimization procedure*' dealing with the separation of the desired polarimetric radar target versus the undesired radar clutter returns of which the alternate lexicographic and Pauli-based covariance matrix optimization procedures deserve special attention next to the coherent $[S]$ and partially coherent $[K]$ matrix cases.

4.1 Formulation of the Mono-Static Radar Optimization Procedure according to Kennaugh for the Coherent Case

Kennaugh was the first to treat the mono-static polarimetric radar optimization procedure (see Fig. 4.1) for optimizing (3.9) according to the BSA formulation

$$\mathbf{E}^s(\mathbf{r}) = [S]\,\mathbf{E}^{i*}(\mathbf{r}) \tag{4.1}$$

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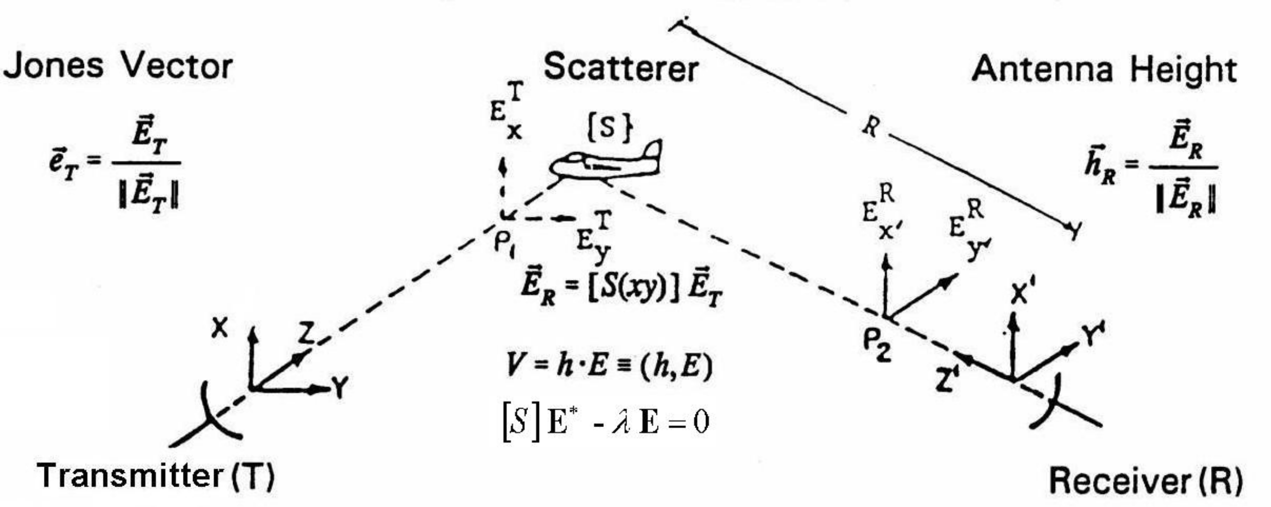


Fig. 4.1 BSA Optimization According to Kennaugh

but with the received field $\mathbf{E}^r(\mathbf{r})$ being so aligned with the incident field $\mathbf{E}^i(\mathbf{r})$ with the reversal of the scattered versus incident coordinates of the BSA system resulting in Kennaugh's psuedo-eigenvalue' [4] problem of

Opt{[S]} such that $[S]\mathbf{E}^* - \lambda\,\mathbf{E} = 0$ (4.2)

The rigorous solution to this set of 'con-similarity eigenvalue' problems was unknown to the polarimetric radar community until the late 1980's, when Lüneburg [54], rediscovering the mathematical tools [116, 117], derived a rigorous but mathematically rather involved method of the associated con-similarity eigenvalue problem, not further discussed here, but we refer to Lüneburg's complete treatment of the subject matter in [52, 53]. Instead, here Chan's [77] *'Three-Step Solution',* as derived from Kennaugh's original work [4], is adopted.

Three Step Procedure according to Chan [77]

By defining the polarimetric radar brightness (polarization efficiency, polarization match factor) formation according to (3.26) and (3.27) retaining the factor $1/2$ (not contained in the 1983 IEEE Standard and in Mott's textbook) [76, 102] as

$$P_R = |V_R|^2 = \frac{1}{2}\left|\mathbf{h}^{r^T}\,[\mathbf{S}]\,\mathbf{E}^i\right|^2 \qquad (4.3)$$

in terms of the terminal voltage V_R, being expressed in terms of the normalized transceiver antenna height $\mathbf{h}^r$ and the incident field $\mathbf{E}^i$, as defined in Mott [76] and in [19], by

$$V_R = \mathbf{h}^{\mathbf{r}^{\mathrm{T}}}\,\mathbf{E}^s = \mathbf{h}^{\mathbf{r}^{\mathrm{T}}}[S]\mathbf{E}^t \quad \textbf{with} \quad \mathbf{h}^{\mathrm{r}} = \frac{\mathbf{E}^{\mathrm{r}}}{\|\mathbf{E}^{\mathrm{r}}\|} \qquad (4.4)$$

so that the total energy density of the scattered wave $\mathbf{E}^s$, may be defined by

$$W = \mathbf{E}^{s^+}\mathbf{E}^s = \left([S]\mathbf{E}^t\right)^+\left([S]\mathbf{E}^t\right) = \mathbf{E}^{t^+}\left([S]^+[S]\right)\mathbf{E}^t = \mathbf{E}^{t^+}[G]\mathbf{E}^t \qquad (4.5)$$

where $[G] = [S]^\dagger[S]$ defines the Graves power density matrix [7], first introduced by Kennaugh [4, 5].

Step 1

Because the solution to the *'pseudo-eigenvalue problem'* of (4.2) was unknown at that time (1954 until 1984); and, since $[S]$ could be, in general, non-symmetric and non-hermitian, Kennaugh embarked instead in determining the *'optimal polarization states'* from optimizing the power density matrix so that

$$[G]\mathbf{E}^{t}_{\mathbf{opt}} - \nu\,\mathbf{E}^{t}_{\mathbf{opt}} = 0 \tag{4.6}$$

for which real positive eigenvalues $\nu_1 = |\lambda_1|^2$ and $\nu_2 = |\lambda_2|^2$ exist for all matrices $[S]$ since $[G]$ is Hermitian positive semidefinite so that

$$\nu_{1,2} = \frac{1}{2}\left\{Trace[G] \pm \left[Trace^2[G] - 4Det[G]\right]^{\frac{1}{2}}\right\} \tag{4.7}$$

where

$$\nu_1 + \nu_2 = \textbf{invariant} = Trace[G] = Span[S] = [S_{HH}]^2 + [S_{HV}]^2 + [S_{VH}]^2 + [S_{VV}]^2 = \kappa_4 \tag{4.8}$$

$$\nu_1 \cdot \nu_2 = \textbf{invariant} = Det[G] = (Det[S])(Det[S])^* = (S_{HH}S_{VV} - S_{HV}S_{VH})(S_{HH}S_{VV} - S_{HV}S_{VH})^* \tag{4.9}$$

For the mono-static, reciprocal symmetric $[S]$, above equations reduce with $S_{HV} = S_{VH}$ to

$$\nu_1 + \nu_2 = \textbf{invariant} = Trace[G] = Span[S] = [S_{HH}]^2 + 2[S_{HV}]^2 + [S_{VV}]^2 = \kappa_3 \tag{4.10}$$

and

$$\nu_1 \cdot \nu_2 = \textbf{invariant} = Det[G] = (Det[S])(Det[S])^* = \left(S_{HH}S_{VV} - |S_{HV}|^2\right)\left(S_{HH}S_{VV} - |S_{HV}|^2\right)^* \tag{4.11}$$

In order to establish the connection between the coneigenvalues of equation (4.2) and the eigenvalues of $[G]$ in (4.6), one may proceed to take the complex conjugate of (4.2) and insert back in (4.2). Equation (4.2) has orthogonal solutions if and only if $[S]$ is symmetric. The inverse step is much more difficult to prove and needs among others the symmetry of $[S]$, which provides another topic for future research.

As a result of these relations, Kennaugh defined the *'effective polarimetric radar cross-section'* ε_{K_4}, also known as '***Kennaugh's Polarimetric Excess*** ε_{K_4} ', in [118], where

$$\varepsilon_{K_4} = Span[S] + 2|Det[S]| \tag{4.12}$$

which comes automatically into play (also in the present formulation) when representations on the Poincare sphere are considered, which reduces to ε_{K_3} for the mono-static reciprocal case. It plays an essential role in Czyz's alternate formulation of the *'theory of radar polarimetry'* [110], derived from a spinorial transformation concept on the *'generalized polarization sphere'*, being studied in more depth by Bebbington [32].

Step 2

Using the resulting solutions for $\nu_{1,2}$ for the known $[G]=[S]^{+}[S]$ and $[S]$, the optimal transmit polarization states $\mathbf{E}^{t}_{\mathbf{opt}1,2}$ and optimal scattered waves $\mathbf{E}^{s}_{\mathbf{opt}1,2}$ can be determined as

$$\mathbf{E}^{s}_{\mathbf{opt}1,2}=[S]\,\mathbf{E}^{t}_{\mathbf{opt}1,2} \tag{4.13}$$

Step 3

The received optimal antenna height $\mathbf{h}^{r}{}_{\mathbf{opt}}$ is then derived from (4.4)

as $$\mathbf{h}^{r}{}_{\mathbf{opt}}=\frac{\mathbf{E}^{s^*}_{\mathbf{opt}}}{\left\|\mathbf{E}^{s}_{\mathbf{opt}}\right\|}=\frac{\left([S]\mathbf{E}^{t}_{\mathbf{opt}}\right)^*}{\left\|[S]\mathbf{E}^{t}_{\mathbf{opt}}\right\|} \tag{4.14}$$

which defines the *'polarization match'* for obtaining maximum power in terms of the polarimetric brightness function (4.4) introduced by Kennaugh in order to solve the polarimetric radar problem [4].

There exist several alternate methods of determining the optimal polarization states either by implementing the *'generalized complex polarization ρ'* transformation, first pursued by Boerner et al. [13]; the *'con-similarity transformation method'* of Lüneburg [52, 53] , the *'spinorial polarization sphere transformations'* of Bebbington [32], and more recently the *'Abelian group method'* of Yang [104 - 105]. It would be worthwhile to scrutinize the various approaches, which should be a topic for future research.

4.2 The Generalized ρ - Transformation for the Determination of the Optimal Polarization States by Boerner using the Critical Point Method

Kennaugh further pioneered the *'polarimetric radar optimization procedures'* by transforming the optimization results on to the polarization sphere, and by introducing the co-polarized versus cross-polarized channel decomposition approach [4] which were implemented but not further pursued by Huynen [9]. Boerner et al. [31, 82], instead, proposed to implement the complex polarization ratio ρ transformation in order to determine the pairs of maximum/minimum back-scattered powers in the co/cross-polarization channels and optimal polarization phase instabilities (cross-polar saddle extrema) by using the *'critical point method'* pioneered in [82]. Assuming that the scattering matrix $[S(HV)]$ is transformed to any other ortho-normal basis {A B} such that

$$\left[S^{'}(AB)\right]=\begin{bmatrix} S^{'}_{AA} & S^{'}_{AB} \\ S^{'}_{BA} & S^{'}_{BB}\end{bmatrix}=[U]^{T}\begin{bmatrix} S_{HH} & S_{HV} \\ S_{VH} & S_{VV}\end{bmatrix}[U] \tag{4.15}$$

with $[U]$ given by (2.23); and $S_{HV}=S_{VH}$, $S^{'}_{AB}=S^{'}_{BA}$ for the mono-static case, the polarimetric radar brightness equation becomes

$$P=\frac{1}{2}\left|V\right|^2=\frac{1}{2}\left|\mathbf{E}^{r^T}[S]\,\mathrm{h}^{t}\right|^2=\frac{1}{2}\left|\mathbf{E}^{r^{'T}}\left[S^{'}\right]\mathrm{h}^{t^{'}}\right|^2 \tag{4.16}$$

where the prime ' refers to any new basis {A B} according to (4.15).

By implementation of the Takagi theorem [116], the scattering matrix $[S^{'}(AB)]$ can be diagonalized [52] so that

$$\left[S'(AB)\right]=\begin{bmatrix} S'_{AA} & 0 \\ 0 & S'_{BB} \end{bmatrix}=\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}=\left[S_d\right] \tag{4.17}$$

$$\lambda_1 = S'_{AA}(\rho_1) = \left(1+\rho_1\rho_1^*\right)^{-1}\left(S_{HH} + 2\rho_1 S_{HV} + \rho_1^2 S_{VV}\right)\ \exp(2j\psi_1) = \left|\lambda_1\right|\ \exp(2j\phi_1) \tag{4.18}$$

$$\lambda_2 = S'_{BB}(\rho_1) = \left(1+\rho_1\rho_1^*\right)^{-1}\left(\rho_1^{*2} S_{HH} - 2\rho_1^* S_{HV} + S_{VV}\right)\ \exp(2j\psi_4) = \left|\lambda_2\right|\ \exp(2j\phi_2) \tag{4.19}$$

as shown in [31].

<u>Determination of the Kennaugh target matrix characteristic polarization states</u>: The expression for the power returned to the co-pol and cross-pol channels of the receiver are determined from the bilinear form to become:

(i) <u>Cross-pol Channel Minima or Nulls (n), Maxima (m) and Saddle-Optima (s)</u>:

For the Cross-pol channel power P_x *with* $\mathbf{E}^r = \mathbf{E}^{t\perp}$, expressed in terms of the antenna length $\mathbf{h}$

$$P_x = \left|V_x\right|^2 = \frac{1}{2}\left|\mathbf{h}'^T_\perp\ [S]\ \mathbf{h}'\right|^2 = \frac{1}{\left(1+\rho'\rho'^*\right)^2}\left(\left|\lambda_1\right|^2\rho'\rho'^* - \lambda_1\lambda_2^*\rho'^{*2} - \lambda_1^*\lambda_2\rho'^2 + \left|\lambda_2\right|^2\rho'\rho'^*\right) \tag{4.20}$$

so that for the cross-pol nulls ($\rho'_{xn1,2}$), for the cross-pol maxima ($\rho'_{xm1,2}$), and for the cross-pol saddle optima ($\rho'_{xs1,2}$), according to the critical point method introduced in [82],

$$\rho'_{xn1,2} = 0,\infty \quad \rho'_{xm1,2} = \pm j\left(\frac{\lambda_1\lambda_2^*}{\lambda_1^*\lambda_2}\right)^{\frac{1}{4}} = \pm\exp(j(2\nu+\pi/2)) \quad \rho'_{xs1,2} = \pm\left(\frac{\lambda_1\lambda_2^*}{\lambda_1^*\lambda_2}\right)^{\frac{1}{4}} = \pm\exp(j(2\nu)) \tag{4.21}$$

where

$$\rho'_{xn1}\rho'^*_{xn2} = -1 \quad \rho'_{xm1}\rho'^*_{xm2} = -1 \quad \rho'_{xs1}\rho'^*_{xs2} = -1 \tag{4.22}$$

which states that there exist three pairs of orthogonal polarization states, the cross-pol minima ($\rho'_{xn1,2}$), the cross-pol maxima ($\rho'_{xm1,2}$), and the cross-pol saddle optima ($\rho'_{xs1,2}$), which are located pair-wise at antipodal points on the polarization sphere so that the lines joining the orthogonal polarization states are at right angles to each other on the polarization sphere[82].

(ii)<u>Co-pol Channel Maxima ($\rho'_{cm1,2}$) and Minima or Nulls ($\rho'_{cn1,2}$):</u>

For the function of the power P_c *with* $\mathbf{E}^r = \mathbf{E}^t$, return to the co-pol channel (c)

$$P_c = \frac{1}{2}\left|V_c\right|^2 = \frac{1}{2}\left|\mathbf{h}'^T\ [\mathrm{S}_d]\ \mathbf{h}'\right|^2 = \frac{1}{\left(1+\rho'\rho'^*\right)^2}\left(\left|\lambda_1\right|^2 + \lambda_1\lambda_2^*\rho'^{*2} + \lambda_1^*\lambda_2\rho'^2 + \left|\lambda_2\right|^2\rho'^2\rho'^{*2}\right) \tag{4.23}$$

the critical points are determined from

$$\rho'_{cm1} = \rho'_{xn1} = 0 \quad \rho'_{cm2} = \rho'_{xn2} = \infty, \quad where \quad \rho'_{cm1}\,\rho'_{cm2} = -1 \tag{4.24}$$

and the co-pol maxima are identical to the cross-pol nulls as was first established by Kennaugh

[4, 5], and utilized by Huynen [9]. In addition the critical points for the co-pol-null minima or nulls ($\rho'_{cn1,2}$) are determined from (4.23) to be

$$\rho'_{cn1,2} = \pm\left(\frac{\lambda_1}{\lambda_2}\right) = \pm\left(\frac{|\lambda_1|}{|\lambda_2|}\right)^{\frac{1}{2}} \exp(j(2\nu + \pi/2)) \qquad (4.25)$$

and it can be shown from above derivations that the co-pol-null minima ρ'_{cn1} and ρ'_{cn2} lie in a plane spanned by the co-pol-maxima (cross-pol-minima) and the cross-pol-maxima and the angle between the origin of the polarization sphere and the two co-pol-nulls is bisected by the line joining the orthogonal pair of co-pol-maxima (cross-pol-minima) defining the target matrix critical angle 2x2γ as shown first by Kennaugh [4] leading to his tensorial polarization fork formulation.

(iii)Orthogonality Conditions with Corresponding Power Returns:

The three pairs of cross-pol-extrema, the cross-pol nulls ($\rho'_{xn1,2}$) being identical to co-pol maxima ($\rho'_{cm1,2}$), the cross-pol maxima ($\rho'_{xm1,2}$) and cross-pol saddle optima ($\rho'_{xs1,2}$), satisfy the orthogonality conditions of (4.22) and (4.24) which implies that they are located each at anti-podal locations on the polarization sphere. We note that the co-pol maxima' consist of one absolute maximum and an orthogonal local maximum. The corresponding co/cross-polar power returns become

$$\begin{aligned}
&Min\{P_x\} = P_{xn1}(\rho'_{xn1}) = P_{xn2}(\rho'_{xn2}) = 0 \;; \\
&Max\{P_c\}: \quad P_{cm1}(\rho'_{cm1}) = |\lambda_1|^2 \qquad P_{cm2}(\rho'_{cm2}) = |\lambda_2|^2 \;; \\
&Max\{P_x\}: \quad P_x(\rho'_{xm1,2}) = \frac{1}{4}(|\lambda_1|^2 + |\lambda_2|^2); \\
&Min\{P_c\}: \quad P_c(\rho'_{cn1,2}) = 0; \\
&Sad\{P_x\}: \quad P_x(\rho'_{xs1,2}) = \frac{1}{4}(|\lambda_1|^2 - |\lambda_2|^2)
\end{aligned} \qquad (4.26)$$

The resulting co/cross-polar extrema are plotted on the polarization sphere shown in Fig. 4.2

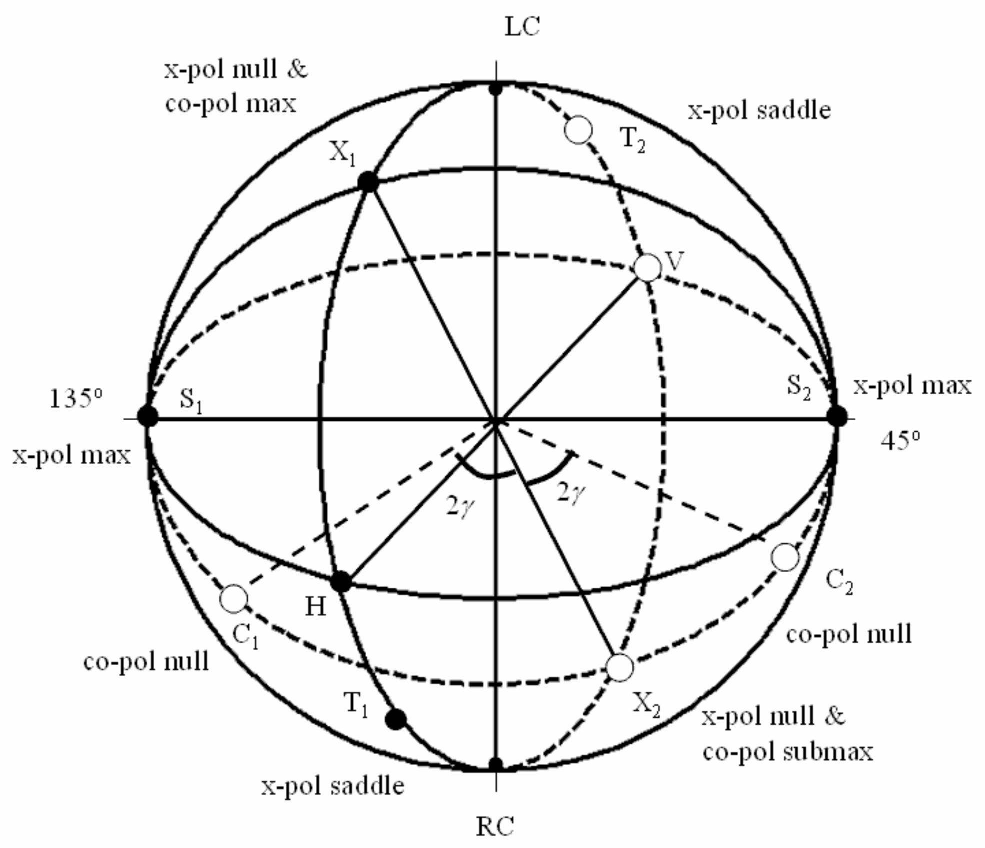


Fig. 4.2 Co/cross-polar Extrema

4.3 The Kennaugh Target Characteristic Scattering Matrix Operator, and the Polarization Fork according to Boerner

Kennaugh was the first to recognize that the orthogonal pairs (X_1, X_2) of the cross-pol nulls ($\rho'_{xn1,2}$) or co-pol maxima ($\rho_{cm1} = \rho_{xn1}, \rho_{cm2} = \rho_{xn2}$) and the pair (S_1, S_2) of cross-pol maxima ($\rho'_{xm1,2}$) lie in one main cross-sectional plane of the polarization sphere also containing the pair (C_1, C_2) of non-orthogonal co-pol nulls ($\rho_{cn1,2}$), where the angle 4γ between the two co-pol null vectors on the Poincaré sphere is bisected by the line joining the two co-pol maxima (cross-pol nulls). These properties were first recognized explicitly and utilized by Kennaugh for defining his "Spinorial Polarization Fork", used later on by Huynen to deduce his *'Polarization Fork'* concept.

However, Boerner et al. [13, 25, 81, 31, 82], by implementing the complex polarization ratio transformation, were able to relate the polarization state coordinates $P(\rho')$ on the Polarization sphere directly to the corresponding ρ' on the complex polarization ratio plane. Then according to [82], each point ρ' of the complex plane can be connected to the *'zenith (LC)'* of the polarization sphere, resting tangent to the complex plane in its *'origin 0'* of the *'nadir (RC)'*, by a straight line that intersects the sphere at one arbitrary point , where the *'nadir (RC)'* corresponds to the *'origin 0'* of the ρ' -plane, the *'zenith (z)'* to the ρ' -circle at *'infinity (∞ '* and the equator representing linear polarization states. Any two orthogonal polarization states are antipodal on the sphere, like *'zenith (left-circular)'*, and *'nadir (right-circular)'*. Utilizing this property, Boerner and Xi [31] were able to associate uniquely three pairs of orthogonal polarization states at right angle (bi-orthogonal) on the polarization sphere; i.e. the anti-podal points S_1, S_2 ($\rho'_{xm1,2}$) and T_1, T_2 ($\rho'_{xs1,2}$), with $\overline{S_1S_2}$ and $\overline{T_1T_2}$ being perpendicular to one another (bi-orthogonal); and similarly to the line $\overline{X_1X_2}$ joining X_1 (nadir: $\rho'_{xn1} = \rho'_{cm1} = 0$) and X_2 (zenith: $\rho'_{xn2} = \rho'_{cm2} = \infty$); where the co-pol nulls ($\rho'_{cn1,2}$) lie on the same main circle on the complex plane of $\rho'_{cn1,2}$ and $\rho'_{xn1,2}$ so that their corresponding points C_1 and C_2 are symmetric about the diameter $\overline{X_1X_2}$ which bisects the angle between C_1, 0, C_2 known as the Kennaugh target matrix characteristic angle (2x2 γ). The *'great cross-sectional plane'* containing $\overline{S_1S_2} \perp \overline{X_1X_2}$ and C_1, C_2 is denoted as the *'Kennaugh target matrix characteristic plane'* with corresponding great circle being the *'Kennaugh target matrix characteristic circle'*. Fig. 4.3 define the representations of the Poincaré sphere for the general and standardized polarization fork (Huynen), respectively, including the proper definitions of *'Huynen's Geometrical Parameters (φ-target orientation angle; γ-target ship angle; τ-target ellipticity angle,* $\rho = \tan\alpha\exp(j\delta)$ *'*, next to *'Kennaugh's target matrix characteristic angle* γ *'.*

This concludes the description of the *'Kennaugh polarimetric target matrix characteristic operator'*; which was coined *'the polarization fork'* by Huynen [9].

4.4 Huynen's Target Characteristic Operator and the Huynen Polarization Fork

Huynen [9], utilizing Kennaugh's prior studies [4, 5], elaborated on polarimetric radar phenomenologies extensively, and his *"Dissertation of 1970: Phenomenological Theory of Radar Targets"* [9], re-sparked international research on *Radar Polarimetry*, commencing with the studies by Poelman [10, 11], Russian studies by Kanareykin [122], Potekhin [123], and others [1].

Huynen cleverly reformulated the definition of the polarization vector, as stated in [9], so that group-theoretic Pauli-spin matrix concepts may favorably be applied which also serve for demonstrating the orientation angle invariance which Huynen coined '*de-psi-ing (de-ψ-ing)*' using ψ for denoting the relative polarization ellipse orientation angle. Here, we prefer to divert from our notation by rewriting the parametric definition of the polarization vector

$$\mathbf{p}(|E|,\phi,\psi,\chi) = |E|\exp(j\phi)\begin{bmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix}\cos\chi\\ -j\sin\chi\end{bmatrix}$$

as

$$\mathbf{p}(a,\alpha,\phi,\tau) = a\exp(j\alpha)\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\tau\\ -j\sin\tau\end{bmatrix} \tag{4.27}$$

which with the use of the Pauli-spin matrices $[\sigma_i]$ defined in (2.14), the Huynen quaternion group definitions may be re-expressed as $[I]=[\sigma_0]$, $[J]=-j[\sigma_3]$, $[K]=j[\sigma_2]$, $[L]=-j[\sigma_1]$

$$\mathbf{p}(a,\alpha,\phi,\tau) = a\exp(j\alpha)\exp(\phi[J])\exp(\tau[K])\begin{bmatrix}1\\0\end{bmatrix} \tag{4.28}$$

with $\exp(\phi[J]) = \cos\phi[I] + \sin\phi[J]$ and $\exp(\tau[K]) = \cos\tau[I] + \sin\tau[K]$

In this notation the orthogonal polarization vector $\mathbf{p}_\perp$ becomes

$$\mathbf{p}_\perp(a_\perp,\alpha_\perp,\phi,\tau) = a_\perp\exp(j\alpha_\perp)\exp\{(\phi+\frac{\pi}{2})[J]\}\exp(-\tau[K])\begin{bmatrix}1\\0\end{bmatrix} \tag{4.29}$$

so that $\mathbf{p}\cdot\mathbf{p}^{*}=a^{2}, \quad \mathbf{p}\cdot\mathbf{p}_{\perp}^{*}=0$.

Utilizing this notation, the transformed matrix $[S^{'}(AB)]$ becomes

$$[S'(AB)]=[U]^{T}[S(HV)][U] \tag{4.30}$$

with the orthonormal transformation matrix $[U]$ defined in (2.23), which may be recasted with $\mathbf{m}=\mathbf{p}_{m}$ the so-called maximum or null polarization as defined in [31], into

$$[U]=[\mathbf{m}\,\mathbf{m}_{\perp}] \tag{4.31}$$

Because of the orthonormal properties of $\mathbf{m}$ and $\mathbf{m}_{\perp}$, which satisfy the con-similarity eigenvalue equation [82], the off-diagonal elements of $[S^{'}(AB)]$ vanish. This in turn can be used to solve for ρ in (2.23), and hence for $\mathbf{m}$ and $\mathbf{m}_{\perp}$, without solving the consimilarity eigenvalue problem of (4.6). The complex eigenvalues $\rho_{xn1,2}$, defined in (4.26) are renamed as $s_{1,2}$ and were defined by Huynen as

$$s_{1}=m\exp\{2j(\upsilon+\beta)\} \qquad s_{2}=m\tan^{2}\gamma\exp\{-2j(\upsilon-\beta)\} \tag{4.32}$$

so that $[S^{'}(AB)]$ of (4.30) becomes

$$[S'(AB)]=[U^{*}(\mathbf{m},\mathbf{m}_{\perp})]\begin{bmatrix} m\exp\{2j(\upsilon+\beta)\} & 0 \\ 0 & m\tan^{2}\gamma\exp\{-2j(\upsilon-\beta)\}\end{bmatrix}[U^{*}(\mathbf{m},\mathbf{m}_{\perp})]^{T} \tag{4.33}$$

where $m=\sigma_{K},\gamma,\psi,\tau_{m},\upsilon,$ *and* β are the Huynen parameters, and m denoting the target matrix magnitude, may be identified to be *"Kennaugh's polarimetric excess* σ_{K} *"* defined in (4.12); and $\mathbf{m}(\psi,\tau_{m})$ may be re-normalized as

$$\mathbf{m}(\psi,\tau_{m})=\begin{bmatrix}\cos\psi & -\sin\psi \\ \sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix}\cos\tau_{m} \\ -j\sin\tau_{m}\end{bmatrix}=\exp(\psi[\mathbf{J}])\exp(\tau_{m}[\mathbf{K}])\begin{bmatrix}1\\0\end{bmatrix} \tag{4.34}$$

and finally with

$$\begin{bmatrix} m\exp\{2j(\upsilon+\beta)\} & 0 \\ 0 & m\tan^{2}\gamma\exp\{-2j(\upsilon-\beta)\}\end{bmatrix}=m\exp(j\beta)\begin{bmatrix}1 & 0\\ 0 & \tan^{2}\gamma\end{bmatrix}\exp(\upsilon[\mathbf{L}]) \tag{4.35}$$

and $[L]$ representing the third modified Pauli-spin matrix, satisfying in Huynen's notation

$$[L]=[J][K]=-[K][J]-\begin{bmatrix}-j & 0\\ 0 & j\end{bmatrix}, \qquad [L]^{2}=-[I] \tag{4.36}$$

we obtain Huynen's target matrix characteristic operator

$$[\mathbf{H}(m,\gamma,\beta;\phi_{m},\upsilon,\tau_{m})]=[U^{*}(\psi,\tau_{m},\upsilon)]\,m\begin{bmatrix}1 & 0\\ 0 & \tan^{2}\gamma\end{bmatrix}[U^{*}(\psi,\tau_{m},\upsilon)]^{T}\exp(j\beta) \tag{4.37}$$

where

$$[U(\psi,\tau_m,\upsilon)]\exp(\psi[\mathbf{J}])\exp(\tau_m[\mathbf{K}])\exp(\upsilon[\mathbf{L}]) \tag{4.38}$$

representing the Eulerian rotations with 2ψ, $2\tau_m$, 2υ about the bi-orthogonal polarization axes $\overline{S_1S_2}$ (connecting the two cross-pol maxima), $\overline{X_1X_2}$ (connecting the two cross-pol nulls, or equivalently, co-pol maxima), and $\overline{T_1T_2}$ (connecting the two saddle optima), respectively, with more detail given in Boerner and Xi [31]. Huynen pointed out the significance of the relative target matrix orientation angle $\Phi = \phi - \psi$, where ϕ denotes the antenna orientation angle, and that from definition of $\mathbf{m}(\psi,\tau_m)$ in (4.34), it can be shown that it can be eliminated from the scattering matrix parameters and incorporated into the antenna polarization vectors (‘*de-psi-ing: de-*ψ *-ing’*) , and that the Huynen parameters are orientation independent, which was more recently analyzed in depth by Pottier [58]. The Eulerian angle are indicators of a scattering matrix’s characteristic structure with υ denoting the so-called ‘*skip-angle*’ related to multi-bounce scattering (single versus double), τ_m denotes the helicity-angle and is an indicator of target symmetry $\tau_m = 0$ or non-symmetry, and β is the absolute phase which is of particular relevance in polarimetric radar interferometry.

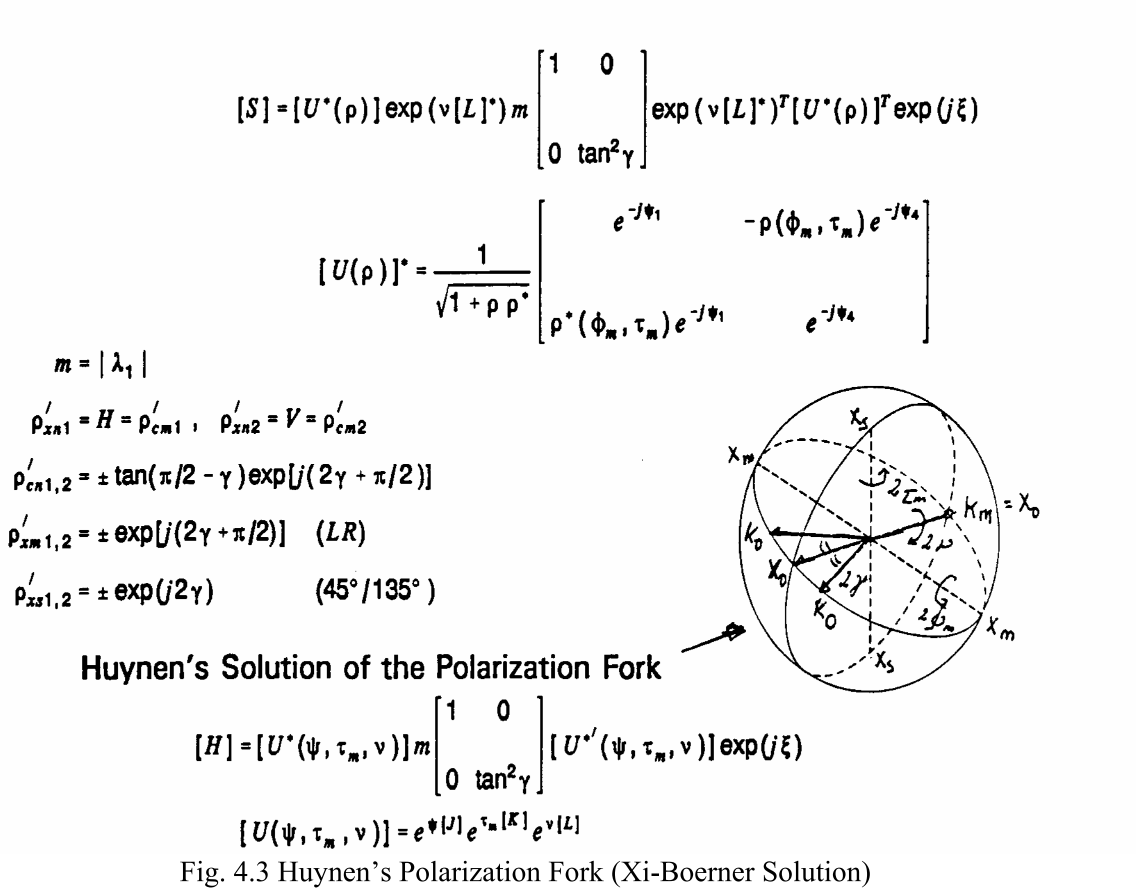


Fig. 4.3 Huynen’s Polarization Fork (Xi-Boerner Solution)

4.5 Alternate Coherent Scattering Matrix Decompositions by Kroggager and by Cameron

Another class of scattering matrix decomposition theorems [124, 30] were recently introduced, and are also expressed in terms of the Pauli spin matrix sets $\psi_p([\sigma_i],\ i = 0,1,2,3)$, by associating elementary scattering mechanisms with each of the $[\sigma_i]$, so that for the general non-symmetric case

$$[S(a,b,c,d)] = \begin{bmatrix} a+b & c-jd \\ c+jd & a-b \end{bmatrix} = a[\sigma_0] + b[\sigma_1] + c[\sigma_2] + d[\sigma_3] \qquad (4.39)$$

where a, b, c and d are all complex. Above '*coherent*' decomposition may be interpreted in terms of **four** '*elementary deterministic point target'* scattering mechanisms, viewed under a change of wave polarization basis, where

a - corresponds to single scattering from a sphere or plane surface
b - corresponds to di-plane scattering (4.40)
c - corresponds to di-plane scattering with a relative orientation of $45°$
d - corresponds to anti-symmetric 'helix-type' scattering mechanisms that transform the incident wave into its orthogonal circular polarization state (helix related)

Krogager [30] and Cameron [124, 125], among others, in essence made use of this decomposition for the symmetric scattering matrix case by selecting the desirable combinations of the $[\sigma_i]$ that suits their specific model cases best.

In the '*Krogager Approach',* a symmetric matrix $[S_K]$ is decomposed into three coherent components, which display the '*physical meaning'* of '*sphere', 'diplane',* and '*helical targets',* where

$$[S_K(a,b,d)] = \alpha\,[S_{sph}] + \mu \exp(j\phi)[S_{di}] + \eta \exp(j\phi)[S_{hel}] \qquad (4.41)$$

with additional direct associations with the Pauli matrices defined in (2.14) given by

$$[S_{sph}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [S_{di}] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad [S_{hel}] = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \qquad (4.42)$$

This decomposition is applied directly to the complex $[S]$ matrix imagery, and results into a rather efficient sorting algorithm in terms of the three characteristic '*feature sorting base scatter images'.* Using color composite presentations for the three classes then allows for the associated *'unsupervised feature sorting'.* This *feature sorting method* has been applied rather successfully in the interpretation of various geo-environmental (forestry, agriculture, fisheries, natural habitats, etc.) as well as in law enforcement and military applications.

In the '*Cameron Approach',* the matrix $[S_C(a,b,c,d)]$ is decomposed, by separating the non-reciprocal $[S_{C_{nr}}]$ from the reciprocal component $[S_{C_{rec}}] = [S_{C_{sym}}]$ via an orientation angle ϕ', and by further decomposing the latter into two further components, $[S_{C_{sym}}^{\max}]$ and $[S_{C_{sym}}^{\min}]$, with linear eigen-polarizations via the angle τ, so that

$$[S_C(a,b,c,d)] = a\cos\phi' \left\{\cos\tau\,[S_{C_{sym}}^{\max}] + \sin\tau\,[S_{C_{sym}}^{\min}]\right\} + a\sin\phi'\,[S_{C_{nr}}] \qquad (4.43)$$

which is further analyzed in [40] and its limitations are clearly identified in [27].

Of the many other existing $[S(a,b,c,d)]$ matrix decomposition theorems, mostly derived from alternate formulations (4.39) of the Pauli spin matrix set $\psi_p([\sigma_i],\ i=0,1,2,3)$, defined in (2.14), the three examples of the Huynen, the Krogager and the Cameron decompositions, it becomes apparent that there exists an infinum of decompositions non of which is unique and all of them are basis dependent and require *a priori* information on the scatterer scenario under investigation. Yet for specific distinct applications all of them may serve a useful purpose which highly superior to any non-polarimetric or partially polarimetric treatment.

(a) San Francisco Image (b) SDH Decomposition

Fig. 4.4 Polarimetric Decompositions: Krogager, 1993. (a) Original San Francisco POLSAR image with RGB color coded by |HH-VV|, |HV| and |HH+VV|, respectively. (b) Sphere (Blue), diplane (Red) and helix (Green) decomposition (SDH) decomposition.

4.6 Kennaugh Matrix Decomposition of Huynen's Matrix Vector Characteristic Operator

The '*Kennaugh target matrix characteristic operator*' can also be derived from the Kennaugh matrix, as was shown by Boerner et al. [78, 90, 31, 82]; using the Lagrangian multiplier method. A more elegant method was recently devised by Pottier in order to highlight the importance of Huynen's findings on the '*target orientation ψ invariance'* for both the Kennaugh and the Lexicographic Covariance matrix representations; and most recently by Yang [119 - 121], who shed more light into the properties of the '*equi-power-loci'* as well as *'equi-correlation-phase-loci'* which deserve careful future attention but will not be further analyzed here.

Instead, we return to Huynen [9], who provided further phenomenological insight into the properties of the Kennaugh matrix $[K]$, by redefining $[S]$ for the symmetric case in terms of the limited set of Pauli spin matrices.

$$[S(a,b,c)] = \begin{bmatrix} a+b & c \\ c & a-b \end{bmatrix} = a[\sigma_0] + b[\sigma_1] + c[\sigma_2] \tag{4.44}$$

so that with the formal relation of $[K]$ with $[S]$, obtained via a Kroenecker product multiplication as

$$[K] = 2\,[A]^{T_{-1}}\ \left([S]\otimes[S^*]\right)[A]^{-1} \tag{4.45}$$

insertion of (4.44) into (4.45) yields, using Huynen's notation

$$[K_\psi]=\begin{bmatrix}(A_0+B_0) & F_\psi & C_\psi & H_\psi \\ F_\psi & (-A_0+B_0) & G_\psi & D_\psi \\ C_\psi & G_\psi & (A_0+B_1) & E_\psi \\ H_\psi & D_\psi & E_\psi & (A_0-B_0)\end{bmatrix} \tag{4.46}$$

where

$$\begin{aligned} |a|^2 &= 2A_0 & ab^* &= C-jD \\ |b|^2 &= B_0+B_1 & bc^* &= E+jF \\ |c|^2 &= B_0-B_1 & ac^* &= H+jG \end{aligned} \tag{4.47}$$

Recognizing that H becomes zero for proper '*de-ψ-ing*' of $[S]$ as defined in $[H]$ of (4.37) by removing the ψ rotational dependence, and inserting it into the antenna descriptors, he was able to redefine his Kennaugh matrix coefficients such that

$$\begin{aligned} H_\psi &= C\sin 2\psi & C_\psi &= C\cos 2\psi \\ G_\psi &= G\cos 2\psi - D\sin 2\psi & E_\psi &= E\cos 4\psi + F\sin 4\psi \\ D_\psi &= G\sin 2\psi + D\cos 2\psi & F_\psi &= -E\sin 4\psi + F\cos 4\psi \end{aligned} \tag{4.48}$$

so that the '*de-ψ-ed*' $[K]$ becomes, by removing the ψ-dependence from $[K]$ and incorporating it into the antenna polarization and Stokes' vectors, respectively, such that

$$[K_{de-\psi}]=\begin{bmatrix}(A_0+B_0) & F & C & H=0 \\ F & (-A_0+B_0) & G & D \\ C & G & (A_0+B_1) & E \\ H=0 & D & E & (A_0-B_0)\end{bmatrix} \tag{4.49}$$

and expressed in terms of the Huynen parameters

$$\begin{aligned} A_0 &= Qf\cos^2 2\tau_m & G &= Qf\sin 4\tau_m \\ B_0 &= Q\left(1+\cos^2 2\gamma - f\cos^2 2\tau_m\right) & B_1 &= Q\left(1+\cos^2 2\gamma - f\left(1+\sin^2 2\tau_m\right)\right) \\ C &= 2Q\cos 2\gamma\cos 2\tau_m & D &= Q\sin^2 2\gamma\sin 4\upsilon\cos 2\tau_m \\ F &= 2Q\cos 2\gamma\sin 2\tau_m & E &= -Q\sin^2 2\gamma\sin 4\upsilon\sin 2\tau_m \\ Q &= m^2\left(8\cos^4\gamma\right)^{-1} & f &= 1-\sin^2 2\gamma\sin^2 s\upsilon \end{aligned} \tag{4.50}$$

Huynen's decomposition was greeted with comprehension, steadily but slowly so are his phenomenological argumentations. We note that its uniqueness is not guaranteed, because it is not basis-independent as was shown by Pottier [126].

4.7 Optimization of the Kennaugh matrix [K] for the Coherent Case Using Stokes' Vector Formulism

Using the Lagrange multiplier method applied to the received power matrices for the mono-static reciprocal case in terms of the Kennaugh (Stokes reflection) matrices $[K_c]$, $[K_x]$, and $[K_m]$ of (3.30) and (3.31), respectively, derived in [113], enables one to determine the characteristic polarization states similar to the generalized ρ-transformation methods.

For simplicity, the transmitted wave incident on the scatterer is assumed to be a completely polarized and normalized wave $\mathbf{q}^t$ so that

$$q_o^t = \left(q_1^{t2} + q_2^{t2} + q_3^{t2}\right)^{1/2} = 1 \qquad (4.51)$$

and re-stating the received power expression defined in (3.26), (3.29), and (3.34) as functions of the optimal Stokes parameters, where

$$\begin{aligned} P_c &= \mathrm{q}^{tT}\,[K_c]\,\mathrm{q}^t = P_c\left(q_1^t, q_2^t, q_3^t\right), \quad P_x = \mathrm{q}^{tT}\,[K_x]\,\mathrm{q}^t = P_x\left(q_1^t, q_2^t, q_3^t\right), \\ P_m &= \mathrm{q}^{tT}\,[K_m]\,\mathrm{q}^t = P_m\left(q_1^t, q_2^t, q_3^t\right) \end{aligned} \qquad (4.52)$$

are subject to the constraint of (4.51).This requirement dictates the use of the method of Lagrangian multipliers to find the extrema of the received powers P_c, P_x, and P_m. Reformulating the equation of constraint to be given by

$$\phi\left(q_1^t, q_2^t, q_3^t\right) = \left(q_1^{t2} + q_2^{t2} + q_3^{t2}\right)^{1/2} - 1 = 0 \qquad (4.53)$$

then the Lagrangian multipliers method for finding the extreme values of any of the three returned power expressions $P_l\left(q_1^t, q_2^t, q_3^t\right)$ results with $l = c, m, x$ in

$$\frac{\partial P_e}{\partial q_i^t} - \mu \frac{\partial \phi}{\partial q_i^t} = 0 \qquad i = 1, 2, 3 \qquad (4.54)$$

For the corresponding ‘*degenerate deterministic*’ (purely coherent) Kennaugh matrix $[K]$, insertion of the corresponding $P_l\left(q_1^t, q_2^t, q_3^t\right)$ into (4.54) results in a set of Galois equations yielding for:

(i) the extreme *‘co-polar channel power P_c’* four solutions: *two maxima* $\rho_{cm1,2}$ which are orthogonal, and *two minima (nulls)* which may in an extreme pathological case be orthogonal (or identical but generally are not); and not one more or not one less of any of these extrema;

(ii) the returned *‘cross-polar channel power P_x’* with six extreme solutions, being the three non-identical pairs of orthogonal polarization states: the *cross-polar maxima* $\rho_{xm1,2}$, the *cross-polar minima* $\rho_{xn1,2}$, and the *cross-polar saddle optima* $\rho_{xs1,2}$; and not one more or not one less of any of these extrema;

(iii) the returned power for the *‘matched antenna case P_m’* yields only two solutions being identical to the *‘co-polar maxima pair $\rho_{cm1,2} = \rho_{xn1,2}$’* and not one more or not one less of any of these extrema.

In summary, $Opt\left\{P_c\left(q_1^t, q_2^t, q_3^t\right)\right\}$ yields always exactly four solutions; $Opt\left\{P_x\left(q_1^t, q_2^t, q_3^t\right)\right\}$ yields always exactly six solutions (or equivalently three “bi-orthogonal” pairs of orthogonal polarization states);

and $Opt\left\{P_m\left(q_1^t, q_2^t, q_3^t\right)\right\}$ yields always two solutions (or equivalently, exactly one pair of orthogonal polarization states). This represents an important result which also holds for the partially polarized case subject to incidence of a completely polarized wave.

A comparison of methods for a coherent scatterer of a specifically given Sinclair matrix $[S]$, with corresponding $[G]$ and $[K]$, plus $[K_c]$, $[K_x]$, and $[K_m]$, analyzed in [113], clearly demonstrates that all of the methods introduced are equivalent.

4.8 Determination of the Polarization Density Plots by van Zyl, and the Polarization Phase Correlation Plots by Agrawal

In radar meteorology, and especially in *'Polarimetric Doppler Radar Meteorology'* the Kennaugh target matrix characteristic operator concept was well received and further developed in the thesis of Agrawal [78]; and especially analyzed in depth by McCormick [127], and Antar [128] because various hydro-meteoric parameters can directly be associated with the Huynen or alternate McCormick parameters. In radar meteorology, the Poincaré sphere visualization of the characteristic polarization states has become commonplace; whereas in wide area SAR remote sensing the co/cross-polarization and Stokes parameter power density plots on the unwrapped planar transformation of the polarization sphere surface, such as introduced independently - *at the same time* - in the dissertations of van Zyl [79] and Agrawal [78], are preferred.

Because of the frequent use of the *'co/cross-polarization power density plots'*, $P_c(\rho), P_{c\perp}(\rho)$ and $P_x(\rho)$; and the equally important but hitherto rarely implemented '*co/cross-polarization phase correlation plots* $R_c(\rho), R_{c\perp}(\rho)$ *and* $R_x(\rho), R_{x\perp}(\rho)$ ', those are here introduced. Following Agrawal [78], who first established the relation between the '*Scattering Matrix Characteristic Operators of Kennaugh and Huynen*' with the *'polarimetric power-density/ phase-correlation plots'*, we obtain for the reciprocal case $S_{AB} = S_{BA}$

$$[C_{3L}] = \langle \mathbf{f}_{3L} \cdot \mathbf{f}_{3L}^{\dagger} \rangle = \begin{bmatrix} \langle |S_{AA}|^2 \rangle & \sqrt{2}\langle S_{AA}S_{AB}^* \rangle & \langle S_{AA}S_{BB}^* \rangle \\ \sqrt{2}\langle S_{AB}S_{AA}^* \rangle & 2\langle |S_{AB}|^2 \rangle & \sqrt{2}\langle S_{AB}S_{BB}^* \rangle \\ \langle S_{BB}S_{AA}^* \rangle & \sqrt{2}\langle S_{BB}S_{AB}^* \rangle & \langle |S_{BB}|^2 \rangle \end{bmatrix} \tag{4.55}$$

re-expressed in terms of the co/cross-polarimetric power density expressions:

$$P_c(\rho) = \langle |S_{AA}|^2 \rangle \qquad P_{c\perp}(\rho) = \langle |S_{BB}|^2 \rangle \qquad P_x(\rho) = \langle |S_{AB}|^2 \rangle = P_{x\perp}(\rho) \tag{4.56}$$

and the co/cross-polarization phase correlation expressions:

$$R_c(\rho) = \langle S_{AA}S_{BB}^* \rangle \qquad R_{c\perp}(\rho) = \langle S_{BB}S_{AA}^* \rangle \qquad R_x(\rho) = \langle S_{AA}S_{AB}^* \rangle \qquad R_{x\perp}(\rho) = \langle S_{BB}S_{AB}^* \rangle \tag{4.57}$$

so that $[C_{L3}(\rho)]$ may be rewritten according to (3.68) – (3.72) as

$$[C_{3L}(\rho)] = \begin{bmatrix} P_c(\rho) & \sqrt{2}\,R_x(\rho) & R_c(\rho) \\ \sqrt{2}\,R_x(\rho)^* & 2P_x(\rho) & \sqrt{2}\,R_x^{\perp}(\rho)^* \\ R_c(\rho)^* & \sqrt{2}\,R_x^{\perp}(\rho) & P_c^{\perp}(\rho) \end{bmatrix} \tag{4.58}$$

satisfying according to (3.70) and (3.71) the following inter-channel and symmetry relations

$$P_c\left(\rho_\perp = -1\big/\rho^*\right) = P_c^\perp(\rho) \qquad \left|R_x\left(\rho_\perp = -1\big/\rho^*\right)\right| = \left|R_x^\perp(\rho)\right|$$

$$P_x\left(\rho_\perp = -1\big/\rho^*\right) = P_x(\rho) \qquad \left|R_c\left(\rho_\perp = -1\big/\rho^*\right)\right| = \left|R_c(\rho)\right| \tag{4.59}$$

and frequently also the degree of polarization $D_p(\rho)$ and the degree of coherency $\mu(\rho)$ in terms of the directly measurable $P_c(\rho), P_{c\perp}(\rho)$ and $P_x(\rho)$; $R_c(\rho), R_{c\perp}(\rho)$ and $R_x(\rho), R_{x\perp}(\rho)$, provided a 'dual-orthogonal, dual-channel measurement system for coherent and partially coherent scattering ensembles is available requiring high-resolution, high channel isolation, high side-lobe reduction, and high sensitivity polarimetric amplitude and phase correlation, where

$$\mu(\rho) = \frac{|R_x(\rho)|}{\sqrt{P_c(\rho)P_x(\rho)}}, \quad D_p(\rho) = \frac{\left\{[P_c(\rho) - P_x(\rho)]^2 + 4|R_x(\rho)|^2\right\}^{1/2}}{(P_c(\rho) + P_x(\rho))}, \quad where \;\; 0 \le \mu(\rho) \le D_p(\rho) \le 1 \tag{4.60}$$

and for coherent (deterministic) scatterers $\mu = D_p = 1$, whereas for completely depolarized scatterers $\mu = D_p = 0$.

The respective power-density profiles and phase-correlation plots are then obtained from the normalized polarimetric radar brightness functions as functions of (ϕ, τ) with $\frac{-\pi}{2} \le \phi \le \frac{\pi}{2}$, $\frac{-\pi}{4} \le \tau \le \frac{\pi}{4}$ so that

$$\begin{aligned}
V_{AA}(\phi,\tau) &= \mathbf{p}^T(\phi,\tau)[S(HV)]\mathbf{p}(\phi,\tau)\\
V_{AB}(\phi,\tau) &= \mathbf{p}_\perp^\dagger(\phi,\tau)[S(HV)]\mathbf{p}(\phi,\tau)\\
V_{BA}(\phi,\tau) &= \mathbf{p}^T(\phi,\tau)[S(HV)]\mathbf{p}_\perp(\phi,\tau)\\
V_{BB}(\phi,\tau) &= \mathbf{p}_\perp^\dagger(\phi,\tau)[S(HV)]\mathbf{p}_\perp(\phi,\tau)
\end{aligned} \tag{4.61}$$

where

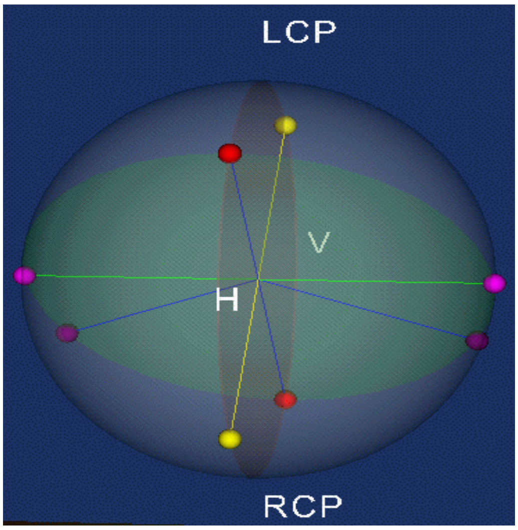
$$\begin{aligned}
P_c &= |V_{AA}|^2 = |S_{AA}(\phi,\tau)|^2\\
P_x &= |V_{AB}|^2 = |S_{AB}(\phi,\tau)|^2\\
R_c &= |\phi_{AA} - \phi_{BB}| = |\arg V_{AA}(\phi,\tau) - \arg V_{BB}(\phi,\tau)|\\
R_x(\phi,\tau) &= |\phi_{AA} - \phi_{AB}| = |\arg V_{AA}(\phi,\tau) - \arg V_{AB}(\phi,\tau)|\\
R_{x\perp}(\phi,\tau) &= |\phi_{BB} - \phi_{BA}| = |\arg V_{BB}(\phi,\tau) - \arg V_{BA}(\phi,\tau)|
\end{aligned} \tag{4.62}$$

In addition, the Maximum Stokes Vector $\mathbf{q}_{0MAX}$, and the maximum received power density P_m may be obtained from

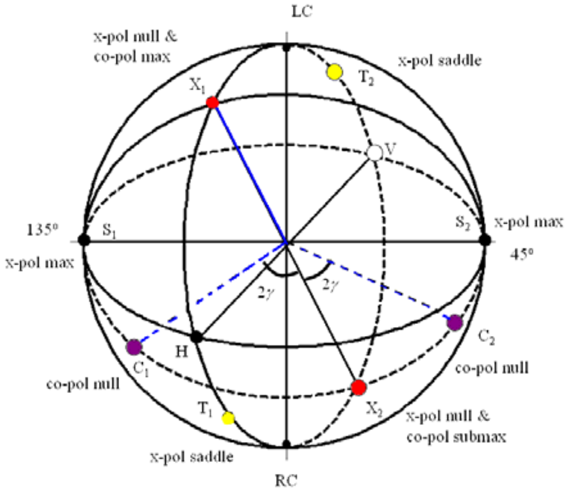
$$P_m(\phi,\tau) = \mathbf{q}_0(\phi,\tau) = [K]\mathbf{q}(\phi,\tau) \tag{4.63}$$

where examples are provided in Figs. 4.5 and 4.6 for one specific matrix case [31, 82] given by

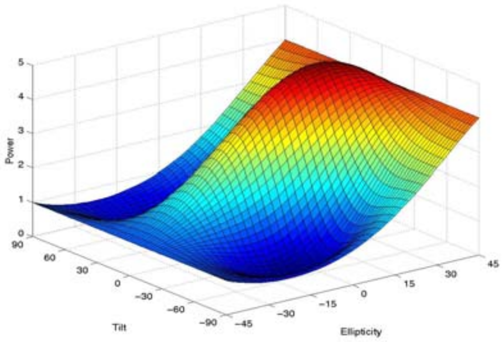
$$[S(HV)] = \begin{bmatrix} 2j & 0.5 \\ 0.5 & -j \end{bmatrix} \tag{4.64}$$



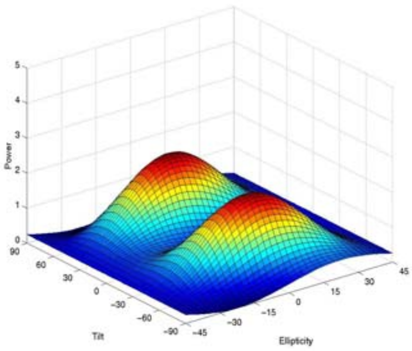
(a) The Kennaugh spinorial (Huynen) polarization fork



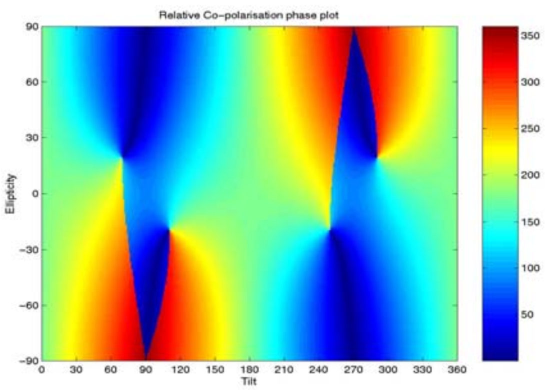
(b) Associated optimal polarization states



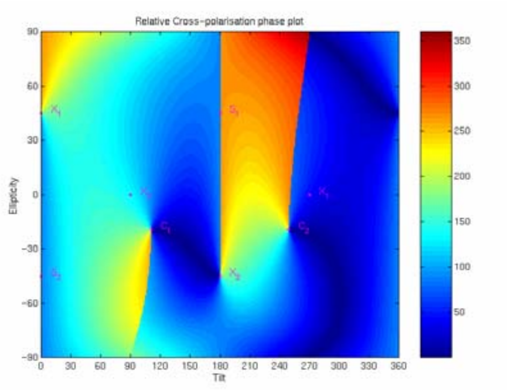
(c) Power density Co-pol



(d) Power density x-pol



(e) Phase correlation co-pol



(f) Phase Correlation x-pol.

Fig. 4.5 Power Density and Phase Correlation Plots for eq. 4.64 (by courtesy of James Morris)

4.9 Optimal Polarization States and its Correspondence to the Density Plots for the Partially Polarized Cases

According to the wave dichotomy portrayed for partially polarized waves, there exists one case for which the coherency matrix for the partially polarized case may be separated into one fully polarized and one completely depolarized component vector according to Chandrasekhar [34]. This principle will here be loosely applied to the case for which a completely polarized wave is incident on either a temporally incoherent (e.g., hydro-meteoric scatter) or spatially incoherent (e.g., rough surface viewed from different depression angles as in synthetic aperture radar imaging). This allows us to obtain a first order approximation for dealing with partially coherent and/or partially polarized waves when the polarimetric entropy is low; for which we then obtain the following optimization criteria:

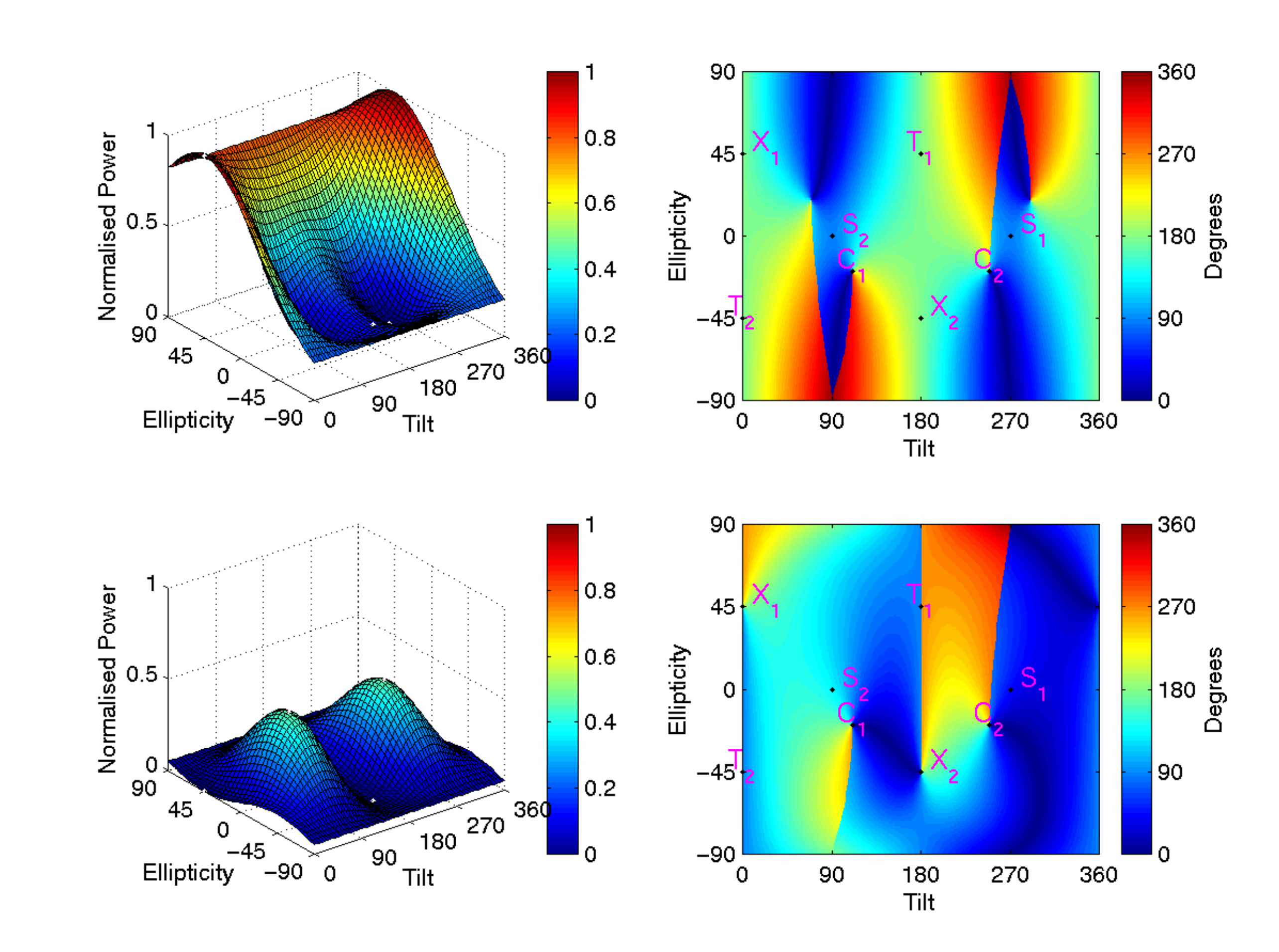


Fig. 4.6 Power Density and Phase Correlation Plots for eq. 4.64 (by courtesy of James Morris)

The energy density arriving at the receiver back-scattered from a distant scatterer ensemble subject to a completely polarized incident wave may be separated into four distinct categories where the Stokes vector is here redefined with $\mathbf{q}_p$ and $\mathbf{q}_u$ denoting the completely polarized and the unpolarized components, respectively

$$\mathbf{q} = \mathbf{q_p} + \mathbf{q_u} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} D_p q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} (1-D_p)q_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.65}$$

and $\mathbf{q}$ as well as D_p were earlier defined, so that the following four categories for optimization of partially polarized waves can be defined as

q_0 total energy density in the scattered wave before it reaches the receiver

$q_0 D_p$ completely polarized part of the intensity

$q_0(1-D_p)$ noise of the unpolarized part

(4.66)

$\frac{1}{2}q_0(1+D_p)$ maximum of the total receptable intensity, the sum of the matched polarized part and one half of the unpolarized part: $\{D_p q_0\} + \left\{\frac{1}{2}(1-D_p)q_0\right\} = \left\{\frac{1}{2}(1+D_p)q_0\right\}$

Considering a time-dependent scatterer which is illuminated by a monochromatic (completely polarized wave) $\mathbf{E}^t$, for which the reflected wave $\mathbf{E}^s$ is, in general, non-monochromatic; and therefore, partially polarized. Consequently, the Stokes vector and Kennaugh matrix formulism will be applied to the four types of energy density terms defined above in (4.66).

4.10 Optimization of the Adjustable Intensity $D_p q_0$

The energy density $D_p q_0$, contained in the completely polarized part $\mathbf{q}_p$ of $\mathbf{q}$, is called the adjustable intensity because one may adjust the polarization state of the receiver to ensure the polarization match as shown previously for the coherent case. We may rewrite the scattering process in index notation as

$$q_i^s = \sum_{j=0}^{3} K_{ij} q_j^t \qquad where \quad j = 0, 1, 2, 3 \tag{4.67}$$

The adjustable intensity $D_p q_0$ can be re-expressed as

$$D_p q_0^s = \left(\sum_{i=0}^{3} q_i^{s2}\right)^{1/2} = \left[\sum_{i=1}^{3}\left(\sum_{j=0}^{3} K_{ij} q_j^t\right)^2\right]^{1/2} \tag{4.68}$$

where the q_i^t are the elements of the Stokes vector of the transmitted wave. The partial derivative of $(D_p q_0)^2$ with respect to q_k^t can be derived as

$$\frac{\partial (D_p q_0^s)^2}{\partial q_k^t} = \sum_{i=1}^{3} \frac{\partial q_i^{s2}}{\partial q_k^t} = 2\sum_{i=1}^{3} q_i^s K_{ik} = 2\sum_{i=1}^{3}\sum_{j=0}^{3} K_{ij} K_{ik} q_j^t \tag{4.69}$$

For optimizing the adjustable intensity, we apply the method of Lagrangian multipliers, which yields

$$\frac{\partial (D_p q_0^s)^2}{\partial q_k^t} - \mu \frac{\partial \phi}{\partial q_k^t} = 2\sum_{i=1}^{3}\sum_{j=0}^{3} K_{ij} K_{ik} q_j^t - \mu q_k^t \tag{4.70}$$

where ϕ is the constraint equation

$$\phi(q_1^t, q_2^t, q_3^t) = (q_1^{t2}, q_2^{t2}, q_3^{t2})^{1/2} - 1 = 0 \tag{4.71}$$

Equation (4.70) subject to (4.71) constitutes a set of inhomogeneous linear equations in $q_1^t(\mu),\ q_2^t(\mu)\ and\ q_3^t(\mu)$, with solutions as three functions of μ. Substituting $q_i^t(\mu; i = 1, 2, 3)$ into the constrained condition (4.71) leads to a sixth-order polynomial Galois equation of μ. For each μ value,

$q_1^t,\ q_2^t,\ q_3^t,\ and\ \ D_p q_0^s$ are calculated according to (4.67) to (4.69). The largest (or smallest) intensity is the optimal intensity; the corresponding $\mathbf{q}^t$ is the optimal polarization state of the transmitted wave.

4.11 Minimizing the Noise-Like Energy Density Term: $q_0^s(1-D_p)$

An unpolarized wave can always be represented by an incoherent sum of any two orthogonal completely polarized waves of equal intensity [14, 15],which leads to 50% efficiency for the reception of the unpolarized wave. In order to receive as much 'polarized energy' as possible, the noise-like energy needs to be minimized. The total energy density of the unpolarized part of the scattered wave is given by:

$$\left(1-D_p\right)q_0^s = q_0^s - D_p q_0^s = \sum_{j=0}^{3} K_{0j} q_j^t - \sqrt{\sum_{i=1}^{3}\left(\sum_{j=0}^{3} K_{ij} q_j^t\right)^2} \tag{4.72}$$

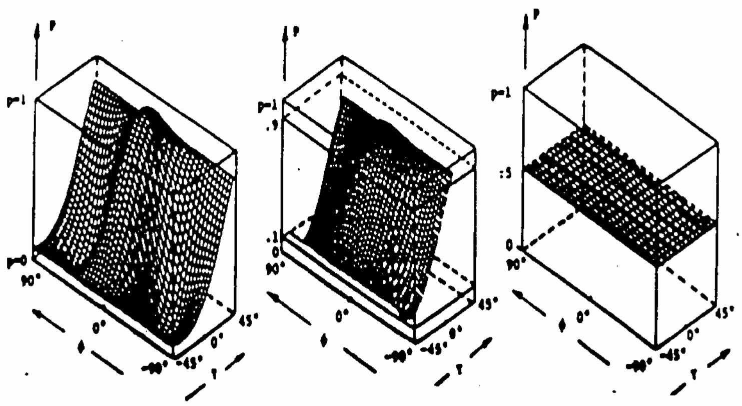
Hitherto, no simple method was found for finding the analytic closed form solution for the minimum; instead, numerical solutions have been developed and are in use.

4.12 Maximizing the Receivable Intensity in the Scattered Wave: $\frac{1}{2} q_0 (1+D_p)$

The total receivable energy density consists of two component parts: 100% reception efficiency for the completely polarized part of the scattered wave and 50% reception efficiency for the unpolarized part. The resulting expression for the total receivable intensity:

$$\frac{1}{2}\left(1+D_p\right)q_0^s = D_p q_0^s + \frac{1}{2}\left(1-D_p\right)q_0^s = \frac{1}{2}\sum_{j=0}^{3} K_{0j} q_j^t + \frac{1}{2}\sqrt{\left(\sum_{i=1}^{3}\sum_{j=0}^{3} K_{ij} q_j^t\right)^2} \tag{4.73}$$

can only be solved using numerical analysis and computation. The resulting maximally received Stokes vector is plotted in Fig. 4.7 (where *q* was replaced by *p*); and we observe that for the fully polarized case no '*depolarization pedestal*' exists. It appears as soon as $p<1$, and for $p=0$ it reaches its maximum of 0.5 for which the polarization diversity profile has deteriorated into the '*flat equal power density profile*', stating that the '*polarization diversity*' becomes meaningless.



p=1
coherent
point scatterer

p = .8
distributed partially
coherent scatterer

p = 0
total polarization
noise

Fig. 4.7 Optimal Polarization States for the Partially Polarized Case

In conclusion, we refer to Boerner et al. [31, 82], where an optimization procedure for (4.65 – 4.68) in terms of $[K]$ for a completely polarized incident wave is presented together with numerical examples. It should be noticed here that Yang more recently provided another more elegant method in [119 - 121] for analyzing the statistical optimization procedure of the Kennaugh matrix.

5. References

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6. Appendices

A. The Standard Kronecker Tensorial Matrix Product

Consider a matrix $[A]=\left[a_{ij}\right]$ of order (*mxn*) and a matrix $[B]=\left[b_{ij}\right]$ of order (*rxs*). The Kronecker product of the two matrices, denoted $[A]\otimes\left[B\right]$ is defined as the partitioned matrix

$$\left[A\right]\otimes\left[B\right]=\begin{bmatrix} a_{11}[B] & a_{12}[B] & \dots & a_{1n}[B] \\ a_{21}[B] & a_{22}[B] & \dots & a_{2n}[B] \\ \vdots & \vdots & & \vdots \\ a_{m1}[B] & a_{m2}[B] & \cdots & a_{mn}[B] \end{bmatrix} \tag{A.1}$$

$[A]\otimes\left[B\right]$ is of order (*mrxns*). It has *mn* blocks; the $(i,j)th$ block is the matrix $a_{ij}[B]$ of order (*r*x*s*).

B. The Mueller Matrix and The Kennaugh Matrix

The Mueller Matrix
For the purely coherent case, the Mueller matrix $[M]$ can formally be related to the coherent Jones scattering matrix $[T]$ as

$$[M]=[1 \quad 1 \quad 1 \quad -1][A]^{T-1}([T]\otimes[T]^*)[A]^{-1}=[A]([T]\otimes[T]^*)[A]^{-1} \qquad \text{(B.1)}$$

with the 4x4 expansion matrix $[A]$ given by:

$$[A]=\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & j & -j & 0 \end{bmatrix}$$

so that the elements M_{ij} of $[M]$ are:

$$\begin{aligned}
M_{11} &= \tfrac{1}{2}\left(|T_{xx}|^2 + |T_{xy}|^2 + |T_{yx}|^2 + |T_{yy}|^2 \right) \\
M_{12} &= \tfrac{1}{2}\left(|T_{xx}|^2 - |T_{xy}|^2 + |T_{yx}|^2 - |T_{yy}|^2 \right) \\
M_{13} &= \operatorname{Re}\left(T_{xx}T_{xy}^* + T_{yx}T_{yy}^* \right) \\
M_{14} &= \operatorname{Im}\left(T_{xx}T_{xy}^* + T_{yx}T_{yy}^* \right) \\
M_{21} &= \tfrac{1}{2}\left(|T_{xx}|^2 + |T_{xy}|^2 - |T_{yx}|^2 - |T_{yy}|^2 \right) \\
M_{22} &= \tfrac{1}{2}\left(|T_{xx}|^2 - |T_{xy}|^2 - |T_{yx}|^2 + |T_{yy}|^2 \right) \\
M_{23} &= \operatorname{Re}\left(T_{xx}T_{xy}^* - T_{yx}T_{yy}^* \right) \\
M_{24} &= \operatorname{Im}\left(T_{xx}T_{xy}^* - T_{yx}T_{yy}^* \right)
\end{aligned}
\qquad
\begin{aligned}
M_{31} &= \operatorname{Re}\left(T_{xx}T_{yx}^* + T_{xy}T_{yy}^* \right) \\
M_{32} &= \operatorname{Re}\left(T_{xx}T_{yx}^* - T_{xy}T_{yy}^* \right) \\
M_{33} &= \operatorname{Re}\left(T_{xx}T_{yy}^* + T_{xy}T_{yx}^* \right) \\
M_{34} &= \operatorname{Im}\left(T_{xx}T_{yy}^* - T_{xy}T_{yx}^* \right) \\
M_{41} &= \operatorname{Im}\left(T_{xx}^*T_{yx} + T_{xy}^*T_{yy} \right) \\
M_{42} &= \operatorname{Im}\left(T_{xx}^*T_{yx} - T_{xy}^*T_{yy} \right) \\
M_{43} &= \operatorname{Im}\left(T_{xx}^*T_{yy} + T_{xy}^*T_{yx} \right) \\
M_{44} &= \operatorname{Re}\left(T_{xx}T_{yy}^* - T_{xy}T_{yx}^* \right)
\end{aligned}
\qquad \text{(B.2)}$$

If $[T]$ is normal, i.e. $[T][T]^{T*}=[T]^{T*}[T]$, then $[M]$ is also normal, i.e. $[M][M]^T=[M]^T[M]$

The Kennaugh Matrix
Similarly, for the purely coherent case, $[K]$ can formally be related to the *coherent Sinclair matrix* $[S]$ with

$[A]^{T-1}=\dfrac{1}{2}[A]^*$ as

$$[K]=2[A]^{T-1}([S]\otimes[S]^*)[A]^{-1} \qquad \text{(B.3)}$$

$$
\begin{aligned}
K_{11} &= \tfrac{1}{2}\left(|S_{xx}|^2 + |S_{xy}|^2 + |S_{yx}|^2 + |S_{yy}|^2 \right) & K_{31} &= \mathrm{Re}\left(S_{xx} S_{yx}^* + S_{xy} S_{yy}^* \right) \\
K_{12} &= \tfrac{1}{2}\left(|S_{xx}|^2 - |S_{xy}|^2 + |S_{yx}|^2 - |S_{yy}|^2 \right) & K_{32} &= \mathrm{Re}\left(S_{xx} S_{yx}^* - S_{xy} S_{yy}^* \right) \\
K_{13} &= \mathrm{Re}\left(S_{xx} S_{xy}^* + S_{yx} S_{yy}^* \right) & K_{33} &= \mathrm{Re}\left(S_{xx} S_{yy}^* + S_{xy}^* S_{yx} \right) \\
K_{14} &= \mathrm{Im}\left(S_{xx} S_{xy}^* + S_{yx} S_{yy}^* \right) & K_{34} &= \mathrm{Im}\left(S_{xx} S_{yy}^* + S_{xy}^* S_{yx} \right) \\
K_{21} &= \tfrac{1}{2}\left(|S_{xx}|^2 + |S_{xy}|^2 - |S_{yx}|^2 - |S_{yy}|^2 \right) & K_{41} &= \mathrm{Im}\left(S_{xx} S_{yx}^* + S_{xy} S_{yy}^* \right) \\
K_{22} &= \tfrac{1}{2}\left(|S_{xx}|^2 - |S_{xy}|^2 - |S_{yx}|^2 + |S_{yy}|^2 \right) & K_{42} &= \mathrm{Im}\left(S_{xx} S_{yx}^* - S_{xy} S_{yy}^* \right) \\
K_{23} &= \mathrm{Re}\left(S_{xx} S_{xy}^* - S_{yx} S_{yy}^* \right) & K_{43} &= \mathrm{Im}\left(S_{xx} S_{yy}^* - S_{yx} S_{xy}^* \right) \\
K_{24} &= \mathrm{Im}\left(S_{xx} S_{xy}^* - S_{yx} S_{yy}^* \right) & K_{44} &= -\mathrm{Re}\left(S_{xx} S_{yy}^* - S_{xy} S_{yx}^* \right)
\end{aligned}
\tag{B.4}
$$

If $[S]$ is symmetric, $S_{xy} = S_{yx}$, then $[K]$ is symmetric, $K_{ij} = K_{ji}$, so that for the symmetric case

$$
\begin{aligned}
K_{11} &= \tfrac{1}{2}\left(|S_{xx}|^2 + 2|S_{xy}|^2 + |S_{yy}|^2 \right) = \tfrac{1}{2} Span[S] \\
K_{12} &= 0 \\
K_{13} &= 0 \\
K_{14} &= 0 \\
K_{21} &= 0 \\
K_{22} &= \tfrac{1}{2}\left(|S_{xx}|^2 - 2|S_{xy}|^2 + |S_{yy}|^2 \right) \\
K_{23} &= \mathrm{Re}\left(S_{xx} S_{xy}^* - S_{yx} S_{yy}^* \right) \\
K_{24} &= 0 \\
K_{31} &= 0 \\
K_{32} &= 0 \\
K_{33} &= |S_{xy}|^2 + \mathrm{Re}\left(S_{xx} S_{yy}^* \right) \\
K_{34} &= 0 \\
K_{41} &= 0 \\
K_{42} &= 0 \\
K_{43} &= 0 \\
K_{44} &= |S_{xy}|^2 - \mathrm{Re}\left(S_{xx} S_{yy}^* \right)
\end{aligned}
\tag{B.5}
$$

with

$$
K_{11} = \sum_{i=2}^{4} K_{ii} = \frac{1}{2}\sum_{i=1}^{4} K_{ii} = \frac{1}{2}\sum_{i=1}^{2} \lambda_i([S]^*[S]) = Span[S]
$$